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On a result of Sullivan and the mod p decomposition of Lie groups.

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§0. Introduction.

Let p be a prime. A simply connected CW complex X is called "mod p decomposable into r spaces" if there exist simply connected CW complexes X_i, $1 \le i \le r$, with $H^*(X_i : Z_p) \neq 0$, and there exists a p-equivalence $f: \prod_{i=1}^{r} X_i \to X$. A mod p decomposition is called irreducible if each X_i is not mod p decomposable.

In the present note, we shall consider the mod p decomposition of SU(n) and other simple Lie groups. For $n \le 2p$ or $n=\infty$, the mod p decomposition of SU(n) has been given by J. P. Serre [9], M. Mimura-H. Toda [6] and F. P. Peterson [8]. Then our result is as follows: Let G be a compact simply connected simply connected simple Lie group. Suppose that $H^*(G)$ has no p-torsion. Hence $H^*(G:Z_p) = \Lambda(x_{n_1}, \ldots x_{n_e})$ is the exterior algebra with deg $x_{n_i} = 2n_i - 1$. Let r(G) be the number of n_i's which are distinct mod p-1.

<u>Main Theorem</u>. Let G be as above and suppose that $H^*(G)$ has no p-torsion. Then G is mod p decomposable irreducibly into r(G) spaces if $G \neq Spin(4n)$. Spin(4n) is mod decomposable irreducibly into r(G)+1 spaces for odd p.

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§1. Localization of CW-complexes.

In this section, we review some results of [3]. Let $\mathbb{P}$ be a subset of all prime numbers. Let $Q_{\mathbb{P}}$ be the ring of the fractions whose denominator, in the lowest term, is prime to p for any $p \in \mathbb{P}$. The void set will be denoted by (0), and hence

$Q \cong Q_{(0)}$ is the field of rational numbers. If $\mathbb{P}$ is void, then a $\mathbb{P}$-equivalence is called a 0-equivalence.

Let $\mathcal{C}_1$ (resp. $\mathcal{F}\mathcal{C}_1$) be the homotopy category of 1-connected (resp. 1-connected with finitely generated homology groups in each dimension) CW-complexes. Then we have

Theorem 1.1 (Theorem 2.4 and 2.5 in [3]). Let $\mathbb{P}$ be a subset of all prime numbers. Then there exists a functor $L_{\mathbb{P}}$: $\mathcal{F}\mathcal{C}_1 \longrightarrow \mathcal{C}_1$ (we denote $L_{\mathbb{P}}(x)$ and $L_{\mathbb{P}}(f)$ briefly by $X_{\mathbb{P}}$ and $f_{\mathbb{P}}$) and a natural indusion $j_X : X \longrightarrow X_{\mathbb{P}}$ satisfying the following conditions.

(i) $f:X \longrightarrow Y$ in $\mathcal{F}\mathcal{C}_1$ is a $\mathbb{P}$-equivalence if and only if $f_{\mathbb{P}}:X_{\mathbb{P}} \longrightarrow Y_{\mathbb{P}}$ is a homotopy equivalence.

(ii) $\pi_*(X_{\mathbb{P}}) \cong \pi_*(X) \otimes Q_{\mathbb{P}}$ and $(j_X)_*:\pi_*(X) \longrightarrow \pi_*(X_{\mathbb{P}})$ coincides $1 \otimes j : \pi_*(X) \otimes Z \longrightarrow \pi_*(X) \otimes Q_{\mathbb{P}}$, where j is the canonical injection.

(iii) $H_*(X_{\mathbb{P}}) \cong H_*(X) \otimes Q_{\mathbb{P}}$ and $(j_X)_*=1\otimes j:H_*(X)\otimes Z \longrightarrow H_*(X)\otimes Q_{\mathbb{P}}$.

For the proof, see [3]. But roughly speaking, the construction of $X_{\mathbb{P}}$ is as follows: For a space X, we associate a direct system $\{X \xrightarrow{f_\lambda} X_\lambda\}$, where f_λ varies all p-equivalences. Then we can define an appropriate linearly ordered cofinal subsystem $\{X_n, f_n\}$ with $X_0=X$, called a $\mathbb{P}$-sequence in [3]. Then $X_{\mathbb{P}}$ is defined by the telescope construction of Adams [1].

Remark D. Sullivan has defined the localization functor for more general category, by use of the Postnikov system.

Now we call a countable CW-complex X finite $\mathbb{P}$-local if $H_*(X)$ is a finitely generated $Q_{\mathbb{P}}$-module.

Theorem 1.2. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be such that $\mathbb{P}_1 \cap \mathbb{P}_2 = (0)$ and $\mathbb{P}_1 \cup \mathbb{P}_2 = \{\text{all primes}\}$. Let $X(\mathbb{P}_i)$ $i=1,2$, be finite $\mathbb{P}_i$-local complexes and let $X(0)$ be a finite 0-local complex. Assume that we are given 0-equivalences $g_i : X(\mathbb{P}_i) \longrightarrow X(0)$. Put $X = X(\mathbb{P}_1) \times_{X(0)} X(\mathbb{P}_2)$, the pull-back of $X(\mathbb{P}_i)$ over $X(0)$. Then X has a homotopy type of a finite CW complex and $X_{\mathbb{P}_i} \simeq X(\mathbb{P}_i)$, $i=1,2$.

For a proof, see [3].

Theorem 1.3. Let $\mathbb{Z}_p$ denote $\mathbb{Z}/p\mathbb{Z}$ if p is a prime and Q if $p=0$. Then $j_X^* : H^*(X_{\mathbb{P}} : \mathbb{Z}_p) \longrightarrow H^*(X : \mathbb{Z}_p)$ is isomorphic if $p \in \mathbb{P}$ or $p=0$. If $p \notin \mathbb{P}$, then $\tilde{H}^*(X_{\mathbb{P}} : \mathbb{Z}_p) \simeq 0$.

Proof. Note that $\mathrm{Hom}(Q_{\mathbb{P}}, \mathbb{Z}_p) \simeq \mathbb{Z}_p$ if $p \in \mathbb{P}$ or $p=0$. and $\mathrm{Hom}(Q_{\mathbb{P}}, \mathbb{Z}_p) \simeq 0$ if $p \notin \mathbb{P}$. Then the theorem immediately follows from the isomorphisms $H^*(X_{\mathbb{P}} : \mathbb{Z}_p) \simeq \mathrm{Hom}(H_*(X_{\mathbb{P}} : \mathbb{Z}_p), \mathbb{Z}_p) \simeq \mathrm{Hom}(H_*(X : \mathbb{Z}_p) \otimes Q_{\mathbb{P}}, \mathbb{Z}_p) \simeq \mathrm{Hom}(H_*(X : \mathbb{Z}_p), \mathrm{Hom}(Q_{\mathbb{P}}, \mathbb{Z}_p))$.

§2. A result of D. Sullivan.

Theorem 2.1. (Sullivan). Let n be an integer and let q be a prime such that $q > n$. Then there exists a map Ψ^q: $BU(n) \longrightarrow BU(n)$ such that $(\Psi^q)^* c_i = q^i c_i$, where $c_i \in H^{2i}(BU(n) : \mathbb{Z})$ is the Chern class.

For a proof, see Chapter 5, of [10].

Now we shall state some easy consequences of Theorem 2.1. First consider the map $\Omega\Psi^q : U(n) \longrightarrow U(n)$. As is well-known $H^*(U(n) : \mathbb{Z}) \simeq \Lambda(h_1, \ldots, h_n)$ is the exterior algebra generated by the universal transgressive generators h_i. Since $(\Psi^q)^* x$

$= q^m x$ for any $x \in H^{2m}(BU(n):Z)$ by Theorem 2.1, we have clearly $(\Omega\Psi^q)^* h_i = q^i h_i$.

Let $k_r : U(n) \longrightarrow U(n)$ be the map defined by $k_r(x) = x^{-q^r}$ for $x \in U(n)$. As is easily checked, $k_r^*(h_i) = -q^r h_i$. We consider the map for $n < q$ and any r

$$\lambda_{q,r} = \Omega\Psi^q + k_r : U(n) \longrightarrow U(n)$$

where the symbol $+$ indicates the sum defined by the multiplication of $U(n)$.

<u>Proposition 2.2</u>. $(\lambda_{q,r})^*(h_i) = (q^i - q^r)h_i$.

<u>Proof</u>. $\lambda_{q,r}$ is defined as the composition

$$U(n) \xrightarrow{\Delta} U(n)\times U(n) \xrightarrow{\Omega\Psi^q \times k_r} U(n)\times U(n) \xrightarrow{\mu} U(n)$$

where Δ is the diagonal map and μ is the multiplication. Then $(\lambda_{q,r})^*(h_i) = \Delta^* \circ (\Omega\Psi^q \times k_r)^* \circ \mu^*(h_i) = \Delta^* \circ (\Omega\Psi^q \times k_r)^*(h_i \otimes 1 + 1 \otimes h_i) = \Delta^*(q^i h_i \otimes 1 + 1 \otimes (-q^r)h_i) = (q^i - q^r)h_i$, since h_i is primitive.

Now notice that we can define a map $\lambda_{q,r}$ on $SU(n)$ satisfying the property of Proposition 2.2. For we have the canonical homeomorphism $SU(n)\times S^1 \cong U(n)$.

§3. Mod p decomposition of $SU(n)$.

In this section, provided with the map $\lambda_{q,r}$, we do the similar arguments in §9 of [3].

<u>Lemma 3.1</u>. Let n be a positive integer and let p be a prime. Then there exists a prime q such that $q > n$ and

q is a primitive root mod p.

Proof. Let k be a primitive root mod p. Then so is $k+pt$ for any positive integer t. Since $(k,p)=1$, there exist infinitely many prime numbers of the form $k+pt$, by the classical theorem of Dirichlet. This proves the lemma.

Lemma 3.2. Let q be a primitive root mod p. Then $q^i-q^r \equiv 0(p)$ if and only if $i-r \equiv 0(p-1)$.

Proof. It is enough to show that $q^m-1 \equiv 0(p)$ if and only if $m \equiv 0(p-1)$. But this is just the definition of the primitive root.

Proposition 3.3. Let n and p be as above. Then for each m such as $2\leq m\leq p$ and $m\leq n$, there exists a 1-connected finite CW-complex $X_m(n)$ and there exists a map $f_m:SU(n) \longrightarrow X_m(n)$ satisfying

(i) $H^*(X_m(n) : \mathbb{Z}_p) \cong \Lambda(x_m, x_{m+p-1}, \ldots, x_{m+s(p-1)})$

where $\deg x_i = 2i-1$ and $s = [\frac{n-m}{p-1}]$ is the largest integer $\leq \frac{n-m}{p-1}$.

(ii) $f_m^*(x_i) = h_i$

Proof. Let q be a prime as in Lemma 3.1. Then by Theorem 2.1 and the remark at the end of §2, we can define $\lambda_{q,r}:SU(n) \longrightarrow SU(n)$ satisfying the property of Proposition 2.2. It is clear that $\lambda_{q,r}$ is a 0-equivalence if r is large enough. Put $g_m=\lambda_{q,(p-1)N+m}$, N large enough and $2\leq m\leq p$.

Then by Proposition 2.2, $g_m^* : H^*(SU(n):\mathbb{Z}_p) \longrightarrow H^*(SU(n):\mathbb{Z}_p)$ satisfies $g_m^* h_i = (q^i - q^m)h_i$ and by Lemma 3.2, $q^i - q^m \equiv 0(p)$ if and only if $i-m \equiv 0(p-1)$.

Now $SU(n)$ is p-universal since it is an H-space, see Theorem 1.7 of [7]. Then by Theorem 5.3 of [3], there exists a p-sequence $\{X_i, h_i\}$ of $SU(n)$ such that $X_i = SU(n)$ for any i, namely $SU(n) \xrightarrow{h_i} SU(n) \xrightarrow{h_2} SU(n) \longrightarrow \cdots$. Then by inserting g_i into the above sequence, infinitely many times in any manner for each $i \neq m$, we have a sequence and the telescope Y_m. Note that this sequence is considered as a "sub" sequence of a 0-sequence of $SU(n)$. Then by taking the telescopes, we have the maps for $2 \leq m \leq p$, [3],

$$SU(n)_{(p)} \xrightarrow{a_m} Y_m \xrightarrow{b_m} SU(n)_{(0)}.$$

It is clear that Y_m is finite p-local. Then by Theorem 1.2, the pull-back $X_m(n) = Y_m \times_{SU(n)_{(0)}} SU(n)_{\mathbb{P}}$, $\mathbb{P} = \{\text{all primes} - p\}$, is a 1-connected CW complex and there exists a map $f_m : SU(n) \longrightarrow X_m(n)$ such that $(f_m)_{(p)} \simeq a_m$. Then by the property of g_i^* and by the definition of Y_m, we have $H^*(Y_m : \mathbb{Z}_p) \cong \Lambda(y_m, y_{m+(p-1)}, \cdots, y_{m+s(p-1)})$, where $\deg y_i = 2i-1$ and $s = [\frac{n-m}{p-1}]$ and $a_m^*(y_i) = h_i$. Then by Theorem 1.2, we can see evidently that $H^*(X_m : \mathbb{Z}_p)$ and f_m^* have the required properties. Q. E. D.

Now put $f = \prod_{m=2}^{p} f_m : SU(n) \longrightarrow \prod_{m=2}^{p} X_m(n)$ with the convention $X_m(n) = *$ and f_m is the trivial map if $m > n$. It is clear by the above proposition that f is a p-equivalence. Since $SU(n)$ is p-universal for any p, there exists a converse p-equivalence

$$g : \prod_{m=2}^{p} X_m(n) \longrightarrow SU(n).$$

Consider the composition $g'_m : X_m(n) \xrightarrow{\iota_m} \prod_{i=2}^{p} X_i(n) \xrightarrow{g}$ $SU(n) \xrightarrow{\pi} SU(n)/SU(m-1)$, where ι_m and π are obvious maps. Clearly $g'^*_m : H^*(SU(n)/SU(m-1) : \mathbb{Z}_p) \longrightarrow H^*(X_m(n) : \mathbb{Z}_p)$ is epimorphic. Thus $X_m(n)$ is the space $B^k_{m-1}(p)$ defined in [6], where $k = [\frac{n-m}{p-1}]+1$. Thus we have obtained:

Theorem 3.4. Let p be a prime and let m, $1 \leq m < p$, be an integer. Then for any integer k, there exists a space $B^k_m(p)$ in [6] and there exists a p-equivalence

$$g : \prod_{m=1}^{p-1} B_m^{k_{n,m}}(p) \longrightarrow SU(n)$$

where $k_{n,m} = [\frac{n-m-1}{p-1}]$.

Corollary 3.5 Let p and m be as above. Then for any integer k, $B^k_m(p)$ is an H-space mod p.

§4. Mod p decomposition of the other classical groups.

In this section, p denotes always an odd prime. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration. A map $s : B \longrightarrow E$ is called a cross-section mod p if $p \circ s$ is a p-equivalence. If E is an H-space mod p and if $p : E \longrightarrow B$ admits a cross-section mod p. Let μ be the multiplication of E. Then if F, E and B are 1-connected finite CW complexes, then $F \times B \xrightarrow{i \times s} E \times E \xrightarrow{\mu} E$ gives a p-equivalence by the Serre's class theory.

Consider the canonical bundles associated classical groups,

$$Sp(n) \longrightarrow SU(2n) \longrightarrow SU(2n)/Sp(n)$$

$$Spin(2n+1) \longrightarrow SU(2n+1) \longrightarrow SU(2n+1)/Spin(2n+1)$$

$$Spin(2n-1) \longrightarrow Spin(2n) \longrightarrow S^{2n-1}.$$

B. Harris [2] has shown that such bundles have cross-section mod p for odd p. Hence we have p-equivalences

$$Sp(n) \times (SU(2n)/Sp(n)) \underset{p}{\sim} SU(2n),$$

$$Spin(2n+1) \times (SU(2n+1)/Spin(2n+1) \underset{p}{\sim} SU(2n+1) \quad \text{and}$$

$$Spin(2n-1) \times S^{2n-1} \underset{p}{\sim} Spin(2n).$$

It is also known [4] that $Sp(n) \underset{p}{\sim} Spin\ (2n+1)$.

<u>Theorem 4.1</u>. Let p be an odd prime. Let $k_{a,b} = [\frac{2(a-b)}{p-1}] + 1$. Then there exist the following p-equivalences

$$Sp(n) \underset{p}{\sim} Spin(2n+1) \underset{p}{\sim} \prod_{m=1}^{\frac{p-1}{2}} B_{2m-1}^{k_{n,m}}(p),$$

$$Spin(2n) \underset{p}{\sim} S^{2n-1} \times \prod_{m=1}^{\frac{p-1}{2}} B_{2m-1}^{k_{n-1,m}}(p),$$

$$SU(2n)/Sp(n) \underset{p}{\sim} \prod_{m=1}^{\frac{p-1}{2}} B_{2m}^{k_{n-1,m}}(p),$$

$$SU(2n+1)/Sp(2n+1) \underset{p}{\sim} \prod_{m=1}^{\frac{p-1}{2}} B_{2m}^{k_{n,m}}(p).$$

Proof is straightforward from Theorem 3.4 and will be left to the reader.

§5. Proof of the Main Theorem.

Theorem 5.1. $SU(n)$ has no mod p decomposition into p spaces. Let p be odd, then $Sp(n)$ and $Spin(2n+1)$ have no mod p decomposition into $\frac{p+1}{2}$ spaces. $Sp(n)$ has no mod 2 decomposition into 2 spaces.

Proof. Assume that $SU(n)$ is mod p decomposable into p spaces, i.e., $\prod_{i=1}^{p} X_i \xrightarrow[p]{f} SU(n)$. It is easy to see that $H^*(X_i:Z_p)$ is an exterior algebra and hence there exists a number t such that the degree of the lowest generator of $H^*(X_t:Z_p)$ is greater than $2p+1$. Let x be such a generator and let $k=\deg x$. Then clearly the mod p Hurewicz homomorphism $h:\pi_k(\Pi X_i)\otimes Z_p \longrightarrow H_k(\Pi X_i:Z_p)$ is non trivial. Hence so is $h:\pi_k(SU(n))\otimes Z_p \longrightarrow H_k(SU(n):Z_p)$. But since $k\geq 2p+1$, this is a contradiction. For $Sp(n)$ and $Spin(2n+1)$, the proof is quite similar. q.e.d.

Proof of Main Theorem. By Theorem 5.1, apparently the mod p decompositions of the classical Lie groups given in Theorem 3.4 and 4.1 are irreducible. Also it is easy to see that the number of spaces in the decompostion is $r(G)$ by the definition of $B_m^k(p)$. For exceptional Lie groups, Main Theorem follows from Theorem 4.2 [6] and Theorem 6.1 [7]. This completes the proof.

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